JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **227,** 251–270 (1998)
ARTICLE NO. AY986106

Some Examples of Conjugate P-Harmonic Differential Forms

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Submitted by Firdaus E. Udwadia

Received April 18, 1997

There has been new interest developing in the study of the Hodge theory and p-harmonic differential forms in recent years, largely pertaining to applications in quasi-conformal analysis, electromagnetic theory, relativity theory, mathematical physics, nonlinear potential theory, etc. In this paper, we develop an effective method to find conjugate p-harmonic differential forms in R^n by applying Hodge theory in calculations with the p-harmonic equation. As applications of the method, we obtain some interesting examples of conjugate p-harmonic differential forms.

1. NOTATION AND DEFINITIONS

Let $e_1, e_2, \dots, e_n$ denote the standard unit basis of $\mathbf{R}^n$. For $l = 0, 1, \dots, n$, the linear space of l-vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \dots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$, is denoted by $\wedge^l = \wedge^l(\mathbf{R}^n)$. The Grassmann algebra $\wedge = \oplus \wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is given by

$$\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$$

with summation over all l-tuples $I = (i_1, i_2, \dots, i_l)$ and all integers $l = 0, 1, \dots, n$.

Throughout this paper, we always assume Ω is a connected open subset of $\mathbf{R}^n$. We write $\mathbf{R} = \mathbf{R}^1$. A differential l-form ω on Ω is a de Rham current (see [12, Chapter III]) on Ω with values in $\wedge^l(\mathbf{R}^n)$. We denote the

0022-247X/98 $25.00

general l-form by $du \in \wedge^l(\mathbf{R}^n)$, where $u \in \wedge^{(l-1)}(\mathbf{R}^n)$ is an $(l-1)$-form. Note that a 0-form is the usual function in $\mathbf{R}^n$. The space of differential l-forms is denoted by $D'(\Omega, \wedge^l)$.

We define the Hodge star operator $\star\colon \wedge \to \wedge$ as follows.

DEFINITION 1. If $\omega = \alpha_{i_1, i_2, \ldots, i_k}(x_1, x_2, \ldots, x_n)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$, $i_1 < i_2 < \cdots < i_k$, is a differential k-form, then $\star\omega = \operatorname{sign}(\pi)\, \alpha_{i_1, i_2, \ldots, i_k}(x_1, x_2, \ldots, x_n)\, dx_{j_1} \wedge \cdots \wedge dx_{n-k}$, where $\pi = (i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$ is a permutation of $(1, \ldots, n)$ and $\operatorname{sign}(\pi)$ is the signature of permutation.

We should notice that the Hodge star operator can be defined equivalently as follows:

DEFINITION 1$'$. If $\omega = \alpha_{i_1, i_2, \ldots, i_k}(x_1, x_2, \ldots, x_n)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} = \alpha_I\, dx_I$, $i_1 < i_2 < \cdots < i_k$, is a differential k-form, then $\star\omega = \star\alpha_{i_1 i_2 \cdots i_k}\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} = (-1)^{\Sigma(I)} \alpha_I\, dx_J$, where $I = (i_1, i_2, \ldots, i_k)$, $J = \{1, 2, \ldots, n\} - I$ and $\Sigma(I) = k(k+1)/2 + \sum_{j=1}^k i_j$.

For example, in $\wedge^1(\mathbf{R}^3)$, $\star dx_1 = (-1)^2\, dx_2 \wedge dx_3 = dx_2 \wedge dx_3$.

We know that the Hodge star operator has the following properties:

(i) $\star$ maps k-forms to $(n-k)$-forms for $0 \le k \le n$.

(ii) Let $e_1, e_2, \ldots, e_n$ be the standard unit basis of $\mathbf{R}^n$ and let $\alpha, \beta \in \wedge$. Then

$$\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n \quad \text{and} \quad \alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta \rangle (\star 1).$$

Hence the norm of $\alpha \in \wedge$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \wedge^0 = \mathbf{R}$. The Hodge star is an isometric isomorphism on $\wedge$ with $\star$ (from $\wedge^l$ to $\wedge^{n-l}$) and $\star\star(-1)^{l(n-l)}$ (from $\wedge^l$ to $\wedge^l$). We denote the exterior derivative by $d\colon D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$ for $l = 0, 1, \ldots, n$. Its formal adjoint operator $d^\star\colon D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \ldots, n$. Note that $d^\star$ is a differential operator and

$$\int_\Omega \langle \alpha, d\beta \rangle\, dv = \int_\Omega \langle d^*\alpha, \beta \rangle\, dv$$

for $\alpha, \beta \in \wedge$. We say that a form ω is closed if and only if $d\omega = 0$ and a form ω is coclosed if and only if $d^\star\omega = 0$.

The beautiful Hodge theory and p-harmonic differential forms have wide applications in many fields, such as quasi-conformal analysis, electromagnetic theory, relativity theory, nonlinear elasticity, mathematical physics, nonlinear potential theory, etc. See, for example, [1], [2], [6], [7], and [10]. Considering the length of the paper, we only mention the key point of Hodge theory. The key point of Hodge theory is that harmonic forms (that is, forms ω such that $d\omega = 0$ and $d^\star\omega = 0$, or equivalently, $(dd^\star + d^\star d)\omega = 0$) defined on a complex manifold are cohomology classes. To be more precise, in every de Rham cohomology class of a complex manifold, there exists a unique harmonic form (this is a theorem similar to the Chevalley–Eilenberg theorem saying that in every de Rham cohomology class of a compact connected Lie group there is a unique bi-invariant form). Also, the following Hodge decomposition theorem plays an important role in the Hodge theory (see [10]).

THEOREM 2. *If $\omega \in L^p(\mathbf{R}^n, \wedge^k)$, $1 < p < \infty$, then there is a $(k-1)$-form α and a $(k+1)$-form β such that*

$$\omega = d\alpha + \delta\beta$$

and $d\alpha + \delta\beta \in L^p(\mathbf{R}^n, \wedge^k)$. Moreover the forms $d\alpha$ and $\delta\beta$ are unique and

$$\alpha \in \operatorname{Ker} \delta \cap L_1^p(R^n, \wedge^{(k-1)}),$$
$$\beta \in \operatorname{Ker} d \cap L_1^p(R^n, \wedge^{(k+1)}),$$

and we have the uniform estimate

$$\|\alpha\|_{L_1^p(R^n)} + \|\beta\|_{L_1^p(R^n)} \le C_p(k,n)\|\omega\|_p$$

for some constant $C_p(k,n)$ independent of ω. Here the Hodge operator δ is defined by $\delta = (-1)^{n(n-k)}\star d\star$, $k = 0, 1, \ldots, n$. The Hodge operator δ is the formal adjoint of d.

Many papers about p-harmonic differential forms have been published recently (see [3–5], [8–11], and [13]). In this paper, we do some calculations with the p-harmonic equation by applying Hodge theory successfully. Hence, we develop an effective method to find conjugate p-harmonic differential forms in $\mathbf{R}^n$. As applications of the method, we obtain some interesting examples of conjugate p-harmonic differential forms, particularly Example 7.

The equation for differential forms

$$d^\star A(x, d\omega) = 0, \tag{1}$$

is called the A-harmonic equation, where A: $\Omega \times \wedge^l(\mathbf{R}^n) \to \wedge^l(\mathbf{R}^n)$ satisfies the following conditions:

$$|A(x,\xi)| \le a|\xi|^{p-1} \tag{$\star$}$$

and

$$\langle A(x,\xi), \xi\rangle \ge |\xi|^p \tag{$\star\star$}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbf{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1). A solution to (1) is an element of the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega \langle A(x, d\omega), d\varphi \rangle \, dv = 0$$

for all $\varphi \in W^1_p(\Omega, \wedge^{l-1})$ with compact support.

In order to formulate the Hardy–Littlewood type estimate it is required first of all that the equation is written in the form of a first order differential system:

$$A(x, du) = d^\star v. \tag{2}$$

In this way we obtain a pair (u, v) of $(l-1)$-form u and $(l+1)$-form v, called the conjugate A-harmonic tensors. For example, $du = d^* v$ is an analogue of a Cauchy–Riemann system in $\mathbf{R}^n$. Clearly, the A-harmonic equation is not affected by adding a closed form to u and coclosed form to v. Therefore, any type of estimates between u and v must be modulo such forms. Suppose that u is a solution to (1) in Ω. Then by the Poincaré lemma, at least locally in a ball B, there exists a form $v \in W^1_q(B), \wedge^{l+1})$, $1/p + 1/q = 1$, such that (2) holds.

DEFINITION 3. When u and v satisfy (2) in Ω, and A^{-1} exists in Ω, we call u and v conjugate A-harmonic differential forms (or conjugate A-harmonic tensors) in Ω.

If u satisfies the p-harmonic equation

$$d^\star\left(du|du|^{p-2}\right) = 0, \tag{3}$$

then

$$du|du|^{p-2} = d^\star v, \tag{4}$$

and v satisfies the conjugate q-harmonic equation

$$d\left(d^\star v|dv|^{q-2}\right) = 0 \tag{5}$$

with $1/p + 1/q = 1$.

DEFINITION 4. If a pair of $(l-1)$-form u and $(l+1)$-form v satisfy (4), then u and v are called conjugate p-harmonic differential forms (or conjugate p-harmonic tensors).

Note that $w = \star v$ satisfies

$$d^{\star}\left(dw|dw|^{q-2}\right) = 0. \tag{6}$$

Also, we should notice that if u is a function, Eq. (3) reduces to the usual p-harmonic equation

$$\operatorname{div}\left(\nabla u|\nabla u|^{p-2}\right) = 0.$$

2. CALCULATIONS WITH THE p-HARMONIC EQUATION

First we develop a method to find conjugate harmonic tensors in $\mathbf{R}^3$ for $p = 2$. Then we consider the general case. If $p = 2$, then Eq. (4) reduces to the following simple form:

$$du = d^{\star} v. \tag{7}$$

Here $u = u(x_1, x_2, x_3)$ is any 0-form (function) and v is a 2-form defined by

$$v = v_1\, dx_1 \wedge dx_2 + v_2\, dx_1 \wedge dx_3 + v_3\, dx_2 \wedge dx_3, \tag{8}$$

where $v_1 = v_1(x_1, x_2, x_3)$, $v_2 = v_2(x_1, x_2, x_3)$, and $v_3 = v_3(x_1, x_2, x_3)$ are functions in $\mathbf{R}^3$. Since $d^{\star} = \star d \star$, we have

$$\begin{aligned} d^{\star} v &= \star d \star v = \star d(\star v) = \star d\left(v_1\, dx_3 - v_2\, dx_2 + v_3\, dx_1\right) \\ &= \star\left(\left(\frac{\partial v_1}{\partial x_1} dx_1 + \frac{\partial v_1}{\partial x_2} dx_2 + \frac{\partial v_1}{\partial x_3} dx_3\right) \wedge dx_3\right. \\ &\quad - \left(\frac{\partial v_2}{\partial x_1} dx_1 + \frac{\partial v_2}{\partial x_2} dx_2 + \frac{\partial v_2}{\partial x_3} dx_3\right) \wedge dx_2 \\ &\quad \left. + \left(\frac{\partial v_3}{\partial x_1} dx_1 + \frac{\partial v_3}{\partial x_2} dx_2 + \frac{\partial v_3}{\partial x_3} dx_3\right) \wedge dx_1 \right). \end{aligned} \tag{9}$$

We know that $dx_i \wedge dx_j = -dx_j \wedge dx_i$ for $i \neq j$ and $dx_i \wedge dx_i = 0$. Hence, we obtain

$$d^{\star} v = \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_3}\right) dx_1 + \left(\frac{\partial v_3}{\partial x_3} - \frac{\partial v_1}{\partial x_1}\right) dx_2 - \left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_3}{\partial x_2}\right) dx_3.$$

Thus, the equation

$$du = d^{\star} v$$

is equivalent to the system

$$\begin{aligned}\frac{\partial u}{\partial x_1} &= \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_3},\\ \frac{\partial u}{\partial x_2} &= \frac{\partial v_3}{\partial x_3} - \frac{\partial v_1}{\partial x_1},\\ \frac{\partial u}{\partial x_3} &= -\frac{\partial v_2}{\partial x_1} - \frac{\partial v_3}{\partial x_2}.\end{aligned} \tag{10}$$

When $q = 2$, the conjugate q-harmonic equation (5) reduces to

$$d(d^\star v) = 0. \tag{11}$$

From (9), we obtain

$$\begin{aligned} d(d^\star v) = {} & \left(-\frac{\partial^2 v_1}{\partial x_2^2} - \frac{\partial^2 v_2}{\partial x_2\,\partial x_3} - \frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_1\,\partial x_3}\right) dx_1 \wedge dx_2\\ & - \left(\frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_1\,\partial x_2} + \frac{\partial^2 v_1}{\partial x_2\,\partial x_3} + \frac{\partial^2 v_2}{\partial x_3^2}\right) dx_1 \wedge dx_3\\ & + \left(-\frac{\partial^2 v_2}{\partial x_1\,\partial x_2} - \frac{\partial^2 v_3}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_1\,\partial x_3} - \frac{\partial^2 v_3}{\partial x_3^2}\right) dx_2 \wedge dx_3. \end{aligned}$$

So the equation $d(d^\star v) = 0$ is equivalent to the following system:

$$\begin{aligned} -\frac{\partial^2 v_1}{\partial x_2^2} - \frac{\partial^2 v_2}{\partial x_2\,\partial x_3} - \frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_1\,\partial x_3} &= 0,\\ \frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_1\,\partial x_2} + \frac{\partial^2 v_1}{\partial x_2\,\partial x_3} + \frac{\partial^2 v_2}{\partial x_3^2} &= 0,\\ -\frac{\partial^2 v_2}{\partial x_1\,\partial x_2} - \frac{\partial^2 v_3}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_1\,\partial x_3} - \frac{\partial^2 v_3}{\partial x_3^2} &= 0. \end{aligned} \tag{12}$$

Thus, in order to find harmonic tensors, we only need to solve system (10). This gives us a method to find conjugate harmonic tensors for the case $p = q = 2$.

Similarly to the case of $p = 2$, we know that the equation

$$du|du|^{p-2} = d^\star v$$

is equivalent to (we use notation $\nabla u = du$)

$$\begin{aligned} \frac{\partial u}{\partial x_1}|\nabla u|^{p-2} &= \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_3}, \\ \frac{\partial u}{\partial x_2}|\nabla u|^{p-2} &= \frac{\partial v_3}{\partial x_3} - \frac{\partial v_1}{\partial x_1}, \\ \frac{\partial u}{\partial x_3}|\nabla u|^{p-2} &= -\frac{\partial v_2}{\partial x_1} - \frac{\partial v_3}{\partial x_2} \end{aligned} \tag{13}$$

with $p \neq 2$. Note

$$\nabla u = du = \frac{\partial u}{\partial x_1}\,dx_1 + \frac{\partial u}{\partial x_2}\,dx_2 + \frac{\partial u}{\partial x_3}\,dx_3.$$

We only need to find a p-harmonic function u satisfying (13) and a 2-form v defined by (8) and satisfying (13). Such a pair of u and v are conjugate p-harmonic tensors. We should note that system (13) has infinitely many solutions.

Now we are going to develop a method to find p-harmonic tensors in $\mathbf{R}^n$. Similarly to the case of $n = 3$, we try to get the equivalent system for the equation

$$du|du|^{p-2} = d^{\star} v$$

in $\mathbf{R}^n$. Let

$$v = \sum_{i<j} v_{ij}\,dx_i \wedge dx_j \tag{14}$$

be a 2-form in $\mathbf{R}^n$, where $v_{ij} = v_{ji}$, $1 \le i, j \le n$. Then

$$\star v = \sum_{i<j} (-1)^{2(2+1)/2+i+j} v_{ij}\,dx_{I-\{i,j\}},$$

where $I = \{1, 2, \ldots, n\}$, that is,

$$\star v = \sum_{i<j} (-1)^{i+j+1} v_{ij}\,dx_{I-\{i,j\}}.$$

Thus

$$d(\star v) = \sum_{i<j} (-1)^{i+j+1}\left((-1)^{i-1}\frac{\partial v_{ij}}{\partial x_i}\,dx_{I-\{j\}} + (-1)^{j-2}\frac{\partial v_{ij}}{\partial x_j}\,dx_{I-\{i\}}\right).$$

Therefore, we have

$$
\begin{aligned}
d^{\star} v &= \star d(\star v)\\
&= \sum_{i<j} (-1)^{i+j+1}(-1)^{n(n-1)/2+n(n+1)/2}\\
&\quad \times\left((-1)^{-j+i-1}\frac{\partial v_{ij}}{\partial x_i}\,dx_j + (-1)^{-i+j}\frac{\partial v_{ij}}{\partial x_j}\,dx_i\right)\\
&= \sum_{i<j}\left((-1)^{n^2}\frac{\partial v_{ij}}{\partial x_i}\,dx_j + (-1)^{n^2+1}\frac{\partial v_{ij}}{\partial x_j}\,dx_i\right)\\
&= (-1)^{n^2}\sum_{i<j}\left(\frac{\partial v_{ij}}{\partial x_i}\,dx_j - \frac{\partial v_{ij}}{\partial x_j}\,dx_i\right)\\
&= (-1)^{n^2}\left(\sum_{i<j}\frac{\partial v_{ij}}{\partial x_i}\,dx_j - \sum_{i<j}\frac{\partial v_{ij}}{\partial x_j}\,dx_i\right)\\
&= (-1)^{n^2}\left(\sum_{j=2}^{n}\left(\sum_{i=1}^{j-1}\frac{\partial v_{ij}}{\partial x_i}\right)dx_j - \sum_{j=2}^{n}\sum_{i=1}^{j-1}\frac{\partial v_{ij}}{\partial x_j}\,dx_i\right).
\end{aligned}
$$

Note $v_{ij} = v_{ji}$, so that the second sum can be written as

$$
\begin{aligned}
-\sum_{j=2}^{n}\sum_{i=1}^{j-1}\frac{\partial v_{ij}}{\partial x_j}\,dx_i &= -\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial v_{ij}}{\partial x_j}\,dx_i\\
&= -\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\frac{\partial v_{ji}}{\partial x_i}\,dx_j\\
&= \sum_{i=2}^{n}\left(-\frac{\partial v_{1i}}{\partial x_i}\,dx_1\right) + \sum_{j=2}^{n-1}\left(\sum_{i=j+1}^{n}\left(-\frac{\partial v_{ji}}{\partial x_i}\right)\right)dx_j.
\end{aligned}
$$

Similarly, we have

$$
\sum_{j=2}^{n}\left(\sum_{i=1}^{j-1}\frac{\partial v_{ij}}{\partial x_i}\right)dx_j = \sum_{i=1}^{n-1}\frac{\partial v_{in}}{\partial x_i}\,dx_n + \sum_{j=2}^{n-1}\left(\sum_{i=1}^{j-1}\frac{\partial v_{ij}}{\partial x_i}\right)dx_j.
$$

Hence, we obtain

$$
\begin{aligned}
d^{\star} v = (-1)^{n^2} \Bigg(& \sum_{i=1}^{n-1} \frac{\partial v_{in}}{\partial x_i}\, dx_n + \sum_{j=2}^{n-1} \sum_{i=1}^{j-1} \frac{\partial v_{ij}}{\partial x_i}\, dx_j \\
& - \sum_{i=2}^{n} \frac{\partial v_{1i}}{\partial x_i}\, dx_1 - \sum_{j=2}^{n-1} \sum_{i=j+1}^{n} \frac{\partial v_{ji}}{\partial x_i}\, dx_j \Bigg) \\
= (-1)^{n^2} \Bigg(& - \sum_{i=2}^{n} \frac{\partial v_{1i}}{\partial x_i}\, dx_1 + \sum_{i=2}^{n-1} \left(\sum_{i=1}^{j-1} \frac{\partial v_{ij}}{\partial x_i} - \sum_{i=j+1}^{n} \frac{\partial v_{ji}}{\partial x_i} \right) dx_j \\
& + \sum_{i=1}^{n-1} \frac{\partial v_{in}}{\partial x_i}\, dx_n \Bigg).
\end{aligned}
$$

By (4), we have the following equation:

$$
\begin{aligned}
du|du|^{p-2} = (-1)^{n^2} \Bigg(& - \sum_{i=2}^{n} \frac{\partial v_{1i}}{\partial x_i}\, dx_1 + \sum_{j=2}^{n-1} \left(\sum_{i=1}^{j-1} \frac{\partial v_{ij}}{\partial x_i} - \sum_{i=j+1}^{n} \frac{\partial v_{ji}}{\partial x_i} \right) dx_j \\
& + \sum_{i=1}^{n-1} \frac{\partial v_{in}}{\partial x_i}\, dx_n \Bigg). \qquad (15)
\end{aligned}
$$

If $u = u(x_1, x_2, \ldots, x_n)$ is a function, then (15) can be written as the following system:

$$
\begin{aligned}
\frac{\partial u}{\partial x_1} |\nabla u|^{p-2} &= (-1)^{n^2} \left(- \sum_{i=2}^{n} \frac{\partial v_{1i}}{\partial x_i} \right), \\
\frac{\partial u}{\partial x_j} |\nabla u|^{p-2} &= (-1)^{n^2} \left(\sum_{i=1}^{j-1} \frac{\partial v_{ij}}{\partial x_i} - \sum_{i=j+1}^{n} \frac{\partial v_{ji}}{\partial x_i} \right), \qquad (16) \\
& \qquad j = 2, 3, \ldots, n-1, \\
\frac{\partial u}{\partial x_n} |\nabla u|^{p-2} &= (-1)^{n^2} \sum_{i=1}^{n-1} \frac{\partial v_{in}}{\partial x_i}.
\end{aligned}
$$

Thus, we know that (4) is equivalent to system (16). In order to find p-harmonic differential forms, we only need to find u and v satisfying (16).

Also, from a result in [10], we obtain the relationship between the K-quasi-regular mappings and the conjugate A-harmonic tensors (for the definition of K-quasi-regularity, see [8] or [10]).

THEOREM 5. *Let $f(x) = (f^1, f^2, \dots, f^n)$ be K-quasi-regular in $\mathbf{R}^n$. Then*

$$u = f^l \, df^1 \wedge df^2 \wedge \cdots \wedge df^{l-1}$$

and

$$v = * f^{l+1} \, df^{l+2} \wedge \cdots \wedge df^n,$$

$l = 1, 2, \dots, n-1$, *are conjugate A-harmonic tensors, where A is some operator satisfying* (★) *and* (★★) *in Section* 1.

3. SOME EXAMPLES OF CONJUGATE A-HARMONIC TENSORS

In this section, we obtain some examples of conjugate A-harmonic tensors by using the methods developed in the last section. It is easy to show that $u = C_1\theta + C_2$ is a p-harmonic function $\mathbf{R}^3$, where $\theta = \arctan(x_2/x_1)$. Using this fact, we have the following example.

EXAMPLE 1. Let $u = C_1\theta + C_2 = C_1 \arctan(x_2/x_1) + C_2$ be a function in $\mathbf{R}^3$ and let v be a 2-form defined by

$$v = \left[-\frac{C_1}{2} \ln\left(x_1^2 + x_2^2\right) + F(x_3) \right] dx_1 \wedge dx_2$$
$$+ G(x_2)\, dx_1 \wedge dx_3 + C_3 \, dx_2 \wedge dx_3,$$

where $F(x_3)$ and $G(x_2)$ are differentiable functions and C_i, $i = 1, 2, 3$, are constants. Then u and v are a pair of conjugate harmonic tensors (for the case $p = q = 2$).

In the above example, $v_1 = -(C_1/2)\ln(x_1^2 + x_2^2) + F(x_3)$, $v_2 = G(x_2)$, and $v_3 = C_3$. Then u, v_1, v_2, and v_3 are solutions of (10), and v_1, v_2, v_3 also satisfy (12). Therefore, u and v are a pair of conjugate harmonic tensors.

EXAMPLE 2. Let u be a 0-form (function) defined in Example 1 and let v be a 2-form defined by

$$v = v_1 \, dx_1 \wedge dx_2 + v_2 \, dx_1 \wedge dx_3 + v_3 \, dx_2 \wedge dx_3,$$

where

$$v_1 = -\frac{C_1}{2} \ln\left(x_1^2 + x_2^2\right) - \int G(x_2)\, dx_2 + F(x_3)$$
$$v_2 = x_3 G(x_2) + H(x_2)$$
$$v_3 = C_3.$$

Here $F(x_3)$, $G(x_2)$, and $H(x_2)$ are differentiable functions, and C_1, C_2, C_3 are constants. By checking that u and v satisfy (10), we know that u and v are a pair of conjugate harmonic tensors.

In the above two examples, we just consider the case of $p = 2$ and $n = 3$ (in $\mathbf{R}^3$). Now, we consider the case $p \neq 2$. By the method developed in the last section, we only need to find u and v such that u and v satisfy (4) or system (13). We write some details in the following example.

EXAMPLE 3. We try to solve system (13). Note

$$\nabla u = du = \frac{\partial u}{\partial x_1}\,dx_1 + \frac{\partial u}{\partial x_2}\,dx_2 + \frac{\partial u}{\partial x_3}\,dx_3. \tag{17}$$

If $u = \arctan(x_2/x_1)$, then

$$|\nabla u| = \left(\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2\right)^{1/2} = \left(x_1^2 + x_2^2\right)^{-1/2}, \tag{18}$$

thus,

$$|\nabla u|^{p-2} = \left(x_1^2 + x_2^2\right)^{1-p/2}.$$

Suppose that $u = \arctan(x_2/x_1)$ and $v_3 = C_3$ (constant), then system (13) reduces to

$$\begin{aligned} -x_2\left(x_1^2 + x_2^2\right)^{-p/2} &= \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_3}, \\ x_1\left(x_1^2 + x_2^2\right)^{-p/2} &= -\frac{\partial v_1}{\partial x_1}, \\ 0 &= \frac{\partial v_2}{\partial x_1}. \end{aligned} \tag{19}$$

Similarly to the case of $p = 2$, we have

$$v_2(x_1, x_2, x_3) = g(x_2, x_3), \qquad g \in C^2(\Omega),$$

and

$$\begin{aligned} v_1(x_1, x_2, x_3) &= -\int x_1\left(x_1^2 + x_2^2\right)^{-p/2} dx_1 \\ &= -\frac{1}{2-p}\left(x_1^2 + x_2^2\right)^{1-p/2} + h(x_2, x_3). \end{aligned}$$

Then

$$\frac{\partial v_1}{\partial x_2} = -x_2\left(x_1^2 + x_2^2\right)^{-p/2} + h'_{x_2}$$

and

$$\frac{\partial v_2}{\partial x_3} = g'_{x_3}.$$

Do substitution in the first equation in system (19) to obtain

$$h'_{x_2} + g'_{x_3} = 0.$$

Hence,

$$h(x_2, x_3) = -\int \frac{\partial g}{\partial x_3}\, dx_2 + F(x_3),$$

where $F(x_3) \in C^2(\Omega)$. Thus, we find a group of solutions to (13) as follows:

$$\begin{aligned} u &= \arctan\frac{x_2}{x_1}, \\ v_1 &= -\frac{1}{2-p}\left(x_1^2 + x_2^2\right)^{1-p/2} + F(x_3) - \int \frac{\partial g(x_2, x_3)}{\partial x_3}\, dx_2, \\ v_2 &= g(x_2, x_3), \\ v_3 &= C_1, \end{aligned} \tag{20}$$

where $F(x_3), g(x_2, x_3) \in C^2(\Omega)$ and C_1 is a constant. That is to say, if

$$v = v_1\, dx_1 \wedge dx_2 + v_2\, dx_1 \wedge dx_3 + v_3\, dx_2 \wedge dx_3,$$

then

$$du|du|^{p-2} = d^{\star} v, \qquad p \neq 2,$$

has a group of solutions u and v defined by (20).

Remark. If we choose a different v_3 at the beginning, we might get some other forms of solutions.

EXAMPLE 4. Let

$$u = \frac{1}{\rho} = \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}},$$

$$v_3 = x_1 + x_2 + x_3.$$

By substituting u and v_3 into system (10), we have

$$\begin{aligned} -\frac{x_1}{\rho^3} &= \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_3}, \\ -\frac{x_2}{\rho^3} &= -\frac{\partial v_1}{\partial x_1} + 1, \\ -\frac{x_3}{\rho^3} &= -\frac{\partial v_2}{\partial x_1} - 1. \end{aligned} \tag{21}$$

Solving system (21), we obtain the following solutions:

$$\begin{aligned} v_1 &= \frac{x_1 x_2}{(x_1^2 + x_2^2)\sqrt{x_1^2 + x_2^2 + x_3^2}} + G(x_3) + x_1, \\ v_2 &= \frac{x_1 x_3}{(x_1^2 + x_2^2)\sqrt{x_1^2 + x_2^2 + x_3^2}} + H(x_2) - x_1, \\ v_3 &= x_1 + x_2 + x_3, \end{aligned} \tag{22}$$

where $G(x_3), H(x_2) \in C^2(\Omega)$. So that, for $p = 2$ and $n = 3$, we know that

$$u = \frac{1}{\rho} = \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}},$$

$$v = v_1\, dx_1 \wedge dx_2 + v_2\, dx_1 \wedge dx_3 + v_3\, dx_2 \wedge dx_3$$

are conjugate harmonic tensors, where v_1, v_2, and v_3 are defined by (22).

EXAMPLE 5. Using the same method as in Example 4, we know that

$$u = \frac{1}{\rho} = \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}},$$

$$v_1 = \frac{x_1 x_2}{(x_1^2 + x_2^2)\sqrt{x_1^2 + x_2^2 + x_3^2}} + G(x_3),$$

$$v_2 = \frac{x_1 x_3}{(x_1^2 + x_2^2)\sqrt{x_1^2 + x_2^2 + x_3^2}} + H(x_2),$$

$$v_3 = F(x_1),$$

$F, G, H \in C^2(\Omega)$, define another pair of conjugate harmonic tensors.

EXAMPLE 6. We can show that

$$u = \rho^{(p-3)/(p-1)} = \left(x_1^2 + x_2^2 + x_3^2\right)^{(p-3)/2(p-1)}$$

is a p-harmonic function in $\mathbf{R}^3$. By computation and then simplification, we have the following results:

$$\frac{\partial u}{\partial x_i} = \frac{p-3}{p-1} x_i \rho^{(p+1)/(1-p)}, \qquad i = 1, 2, 3,$$

$$\nabla u = du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \frac{\partial u}{\partial x_3} dx_3,$$

$$|\nabla u| = \left(\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2\right)^{1/2} = \left|\frac{p-3}{p-1}\right| \rho^{2/(1-p)},$$

$$|\nabla u|^{p-2} = \left|\frac{p-3}{p-1}\right|^{p-2} \left(x_1^2 + x_2^2 + x_3^2\right)^{(p-2)/(1-p)}.$$

Therefore, system (13) becomes

$$\begin{aligned} -\left|\frac{p-3}{p-1}\right|^{p-2} x_1\left(x_1^2 + x_2^2 + x_3^2\right)^{(5p-7)/2(1-p)} &= \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_3}, \\ -\left|\frac{p-3}{p-1}\right|^{p-2} x_2\left(x_1^2 + x_2^2 + x_3^2\right)^{(5p-7)/2(1-p)} &= -\frac{\partial v_1}{\partial x_1} + \frac{\partial v_3}{\partial x_3}, \\ -\left|\frac{p-3}{p-1}\right|^{p-2} x_3\left(x_1^2 + x_2^2 + x_3^2\right)^{(5p-7)/2(1-p)} &= -\frac{\partial v_2}{\partial x_1} - \frac{\partial v_3}{\partial x_2}. \end{aligned} \tag{23}$$

Choosing $v_3 = x_1 + x_2 + x_3$ and applying (23), we obtain

$$\begin{aligned} v_1 &= x_1 + a x_2 F(x_1, x_2, x_3) + g(x_2, x_3), \\ v_2 &= -x_1 a x_3 F(x_1, x_2, x_3) + h(x_2, x_3), \\ v_3 &= x_1 + x_2 + x_3, \end{aligned} \tag{24}$$

where

$$F(x_1, x_2, x_3) = \int \left(x_1^2 + x_2^2 + x_3^2\right)^{(5p-7)/2(1-p)} dx_1$$

and $a = -|(p-3)/(p-1)|^{p-2}$, $g, h \in C^1(\Omega)$, satisfy

$$-a x_1\left(x_1^2 + x_2^2 + x_3^2\right)^{(5p-7)/2(1-p)} = a\left(2F + x_2 F'_{x_2} + x_3 F'_{x_3}\right) + g'_{x_2} + h'_{x_3}.$$

Therefore, we have the following conjugate p-harmonic tensors:

$$u = \rho^{(p-3)/(p-1)} = \left(x_1^2 + x_2^2 + x_3^2\right)^{(p-3)/2(p-1)},$$
$$v = v_1\, dx_1 \wedge dx_2 + v_2\, dx_1 \wedge dx_3 + v_3\, dx_2 \wedge dx_3,$$

where v_1, v_2, and v_3 are defined by (24).

For $p = 2$ in $\mathbf{R}^3$, we obtain the following nice example which has beautiful symmetries.

EXAMPLE 7. Let

$$u = \frac{3}{\rho} = \frac{3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

be a harmonic function in $\mathbf{R}^3$. We just switch the positions of v_1 and v_3 in the definition of v in the last example and let v be a 2-form in $\mathbf{R}^3$ defined by

$$v = v_3\, dx_1 \wedge dx_2 + v_2\, dx_1 \wedge dx_3 + v_1\, dx_2 \wedge dx_3.$$

Then (10) becomes

$$\begin{aligned} \frac{\partial u}{\partial x_1} &= \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}, \\ \frac{\partial u}{\partial x_2} &= \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \\ \frac{\partial u}{\partial x_3} &= -\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \end{aligned} \tag{10$'$}$$

and (13) becomes

$$\begin{aligned} \frac{\partial u}{\partial x_1}|\nabla u|^{p-2} &= \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}, \\ \frac{\partial u}{\partial x_2}|\nabla u|^{p-2} &= \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \\ \frac{\partial u}{\partial x_3}|\nabla u|^{p-2} &= -\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}. \end{aligned} \tag{13$'$}$$

Here v_1, v_2, and v_3 are defined as follows:

$$v_1 = \frac{x_2 x_3}{\sqrt{\sum x_i^2}} \frac{x_2^4 - x_3^4}{\prod_{i<j}\left(x_i^2 + x_j^2\right)} = \frac{x_2 x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \frac{x_2^2 - x_3^2}{\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_3^2\right)},$$

$$v_2 = \frac{x_1 x_3}{\sqrt{\sum x_i^2}} \frac{x_1^4 - x_3^4}{\prod_{i<j}\left(x_i^2 + x_j^2\right)} = \frac{x_1 x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \frac{x_1^2 - x_3^2}{\left(x_1^2 + x_2^2\right)\left(x_2^2 + x_3^2\right)},$$

$$v_3 = \frac{x_1 x_2}{\sqrt{\sum x_i^2}} \frac{x_1^4 - x_2^4}{\prod_{i<j}\left(x_i^2 + x_j^2\right)} = \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \frac{x_1^2 - x_2^2}{\left(x_1^2 + x_3^2\right)\left(x_2^2 + x_3^2\right)}.$$

Then u and v are a pair of conjugate harmonic tensors. Now, we check that u, v_1, v_2, and v_3 defined above satisfy $(10)'$. For $u = 3/\rho$, system $(10)'$ reduces to

$$\begin{aligned}
\frac{-3x_1}{\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}} &= \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}, \\
\frac{-3x_2}{\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}} &= \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \\
\frac{-3x_3}{\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}} &= -\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.
\end{aligned} \tag{10$''$}$$

By a long computation and simplification, we have

$$\frac{\partial v_1}{\partial x_2} = \frac{x_3\left(x_2^2 - x_3^2\right)}{\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}} + \frac{2x_3 x_2^2}{\left(x_1^2 + x_2^2\right)^2\left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}},$$

$$\frac{\partial v_1}{\partial x_3} = \frac{x_2\left(x_2^2 - x_3^2\right)}{\left(x_1^2 + x_3^2\right)\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}} + \frac{-2x_2 x_3^2}{\left(x_1^2 + x_3^2\right)^2\left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}},$$

$$\frac{\partial v_2}{\partial x_1} = \frac{x_3\left(x_1^2 - x_3^2\right)}{\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}} + \frac{2x_3 x_1^2}{\left(x_1^2 + x_2^2\right)^2\left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}},$$

$$\frac{\partial v_2}{\partial x_3} = \frac{x_1\left(x_1^2 - x_3^2\right)}{\left(x_2^2 + x_3^2\right)\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}} + \frac{-2x_1 x_3^2}{\left(x_2^2 + x_3^2\right)^2\left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}},$$

$$\frac{\partial v_3}{\partial x_1} = \frac{x_2\left(x_1^2 - x_2^2\right)}{\left(x_1^2 + x_3^2\right)\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}} + \frac{2x_2 x_1^2}{\left(x_1^2 + x_3^2\right)^2\left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}},$$

$$\frac{\partial v_3}{\partial x_2} = \frac{x_1\left(x_1^2 - x_2^2\right)}{\left(x_2^2 + x_3^2\right)\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}} + \frac{-2x_1 x_2^2}{\left(x_2^2 + x_3^2\right)^2\left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}}.$$

Thus, it follows that

$$\begin{aligned}
&\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} \\
&\quad = \frac{x_1\left(2x_1^2 - \left(x_2^2 + x_3^2\right)\right)}{\left(x_2^2 + x_3^2\right)\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}} - \frac{2x_1}{\left(x_2^2 + x_3^2\right)\left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}} \\
&\quad = \frac{-3x_1\left(x_2^2 + x_3^2\right)}{\left(x_2^2 + x_3^2\right)\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}} \\
&\quad = \frac{-3x_1}{\left(x_1^2 + x_2^2 + x_3^2\right)^{3/2}}.
\end{aligned}$$

Hence, the first equation in system $(10)''$ holds. By the same method, we can check that the second and third equations in $(10)''$ hold too.

By a long computation and simplification, we know that

$$\frac{\partial v_1}{\partial x_1} - \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0.$$

So v is a closed 2-form.

EXAMPLE 8. Clearly, $u = a_1x_1 + a_2x_2 + a_3x_3$ is a p-harmonic function in $\mathbf{R}^3$. Here a_i is a constant for $i = 1, 2, 3$. By simple calculation, we have

$$|\nabla u|^{p-2} = |a_1^2 + a_2^2 + a_3^2|^{p/2-1}.$$

If we write $a = |a_1^2 + a_2^2 + a_3^2|^{p/2-1}$, then $(13)'$ reduces to

$$\begin{aligned}
a_1 a &= \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_3}, \\
a_2 a &= \frac{\partial v_3}{\partial x_3} - \frac{\partial v_1}{\partial x_1}, \\
a_3 a &= -\frac{\partial v_2}{\partial x_1} - \frac{\partial v_3}{\partial x_2}.
\end{aligned} \tag{25}$$

(i) If we choose $v_1 = 0$, then, solving the second and third equations in (25), we have

$$\begin{aligned}
v_2 &= -a_3 a x_1 + g(x_2, x_3), \\
v_3 &= -a_2 a x_1 + h(x_2, x_3).
\end{aligned}$$

We may put $g(x_2, x_3) = h(x_2, x_3) = 0$. Thus, we get

$$v_1 = 0,$$
$$v_2 = -a_3 a x_1,$$
$$v_3 = -a_2 a x_1.$$

(ii) Similarly to case (i), now putting $v_2 = 0$, we have

$$v_1 = -a_3 a x_2,$$
$$v_2 = 0,$$
$$v_3 = a_1 a x_2.$$

(iii) In the same way, we obtain

$$v_1 = a_2 a x_3,$$
$$v_2 = a_1 a x_3,$$
$$v_3 = 0.$$

Combining cases (i), (ii), and (iii), we obtain

$$v_1 = \tfrac{1}{2}a(a_2 x_3 - a_3 x_2),$$
$$v_2 = \tfrac{1}{2}a(a_1 x_3 - a_3 x_1),$$
$$v_3 = \tfrac{1}{2}a(a_1 x_2 - a_2 x_1)$$

satisfies (25). Thus, we know that

$$u = a_1 x_1 + a_2 x_2 + a_3 x_3,$$
$$v = \frac{a}{2}(a_1 x_2 - a_2 x_1)\, dx_1 \wedge dx_2 + \frac{a}{2}(a_1 x_3 - a_3 x_1)\, dx_1 \wedge dx_3 + \frac{a}{2}(a_2 x_3 - a_3 x_2)\, dx_2 \wedge dx_3$$

are a pair of conjugate p-harmonic tensors in $\mathbf{R}^3$ for $1 < p < \infty$ and $p \neq 2$.

By a discussion similar to Example 8, we have the following Example 9.

EXAMPLE 9. Let

$$u = \arctan\left(\frac{x_2}{x_1}\right),$$
$$v = \frac{1}{2(p-2)}\left(x_1^2 + x_2^2\right)^{1-p/2} dx_1 \wedge dx_2 - \frac{1}{2} x_2 x_3 \left(x_1^2 + x_2^2\right)^{-p/2} dx_1 \wedge dx_3 + \frac{1}{2} x_1 x_3 \left(x_1^2 + x_2^2\right)^{-p/2} dx_2 \wedge dx_3.$$

Then u and v are a pair of conjugate p-harmonic tensors in $\mathbf{R}^3$.

As we know, $u = \arctan(x_2/x_1)$ is a p-harmonic function for any $p > 1$. So that

$$u = \arctan\frac{x_2}{x_1} + \arctan\frac{x_3}{x_2} + \arctan\frac{x_1}{x_3}$$

is also a p-harmonic function since a p-harmonic equation is symmetric with respect to x_1, x_2, and x_3. By the same method in Example 8, we obtain the following v_1, v_2, and v_3 for $p = 2$:

$$\begin{aligned} v_1 &= \frac{x_1 x_3}{2}\left(x_1^2 + x_2^2\right)^{-1} - \frac{1}{2}\ln\left(x_2^2 + x_3^2\right) + \frac{x_1 x_2}{2}\left(x_1^2 + x_3^2\right)^{-1}, \\ v_2 &= \frac{1}{2}\ln\left(x_1^2 + x_3^2\right) - \frac{x_2 x_3}{2}\left(x_1^2 + x_2^2\right)^{-1} - \frac{x_1 x_2}{2}\left(x_2^2 + x_3^2\right)^{-1}, \qquad (26) \\ v_3 &= \frac{x_1 x_3}{2}\left(x_2^2 + x_3^2\right)^{-1} + \frac{x_2 x_3}{2}\left(x_1^2 + x_3^2\right)^{-1} - \frac{1}{2}\ln\left(x_1^2 + x_2^2\right). \end{aligned}$$

EXAMPLE 10. Let

$$u = \arctan\frac{x_2}{x_1} + \arctan\frac{x_3}{x_2} + \arctan\frac{x_1}{x_3}$$

and let v_1, v_2, v_3 be defined by (26). Then u and v defined by

$$v = v_3\, dx_1 \wedge dx_2 + v_2\, dx_1 \wedge dx_3 + v_1\, dx_2 \wedge dx_3$$

are a pair of conjugate harmonic tensors in $\mathbf{R}^3$.

ACKNOWLEDGMENTS

I would like to thank Professor Craig A. Nolder for his advice and continuous encouragement in the preparation of this paper. The author also thanks the referee and the editor, Professor Firdaus E. Udwadia, for their precious and thoughtful suggestions on this paper.

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